Summing these equations we have

$$[r^2] = r^2 \{ m-[a^2][\alpha^2]-[ab][\alpha\beta]-[ac][\alpha\gamma]-\ldots-[al][\alpha\lambda]$$
$$-[ab][\alpha\beta]-[b^2][\beta^2]-[bc][\beta\gamma]-\ldots-[bl][\beta\lambda]$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$-[al][\alpha\lambda]-[bl][\beta\lambda]-[cl][\gamma\lambda]-\ldots-[l^2][\lambda^2] \}$$
$$= r^2(m-n), \text{ by virtue of (58), (58')}. \qquad (60'')$$

But as already stated the series of corrections, $\varDelta_1, \varDelta_2, \varDelta_3, \ldots\ldots \varDelta_m$, conforms to the laws of error as near as the conditions of the problem permit, therefore we may consider it an ideal series. For such a series and not for a series of *true errors* (as most authors do) we can employ formula (53) to obtain its probable error. We have evidently, for an ideal series,

$$[r^2] = mr^2_{\triangle};$$
$$\therefore\ mr^2_{\triangle} = (m-n)r^2. \qquad (60''')$$

We have then the remarkable relation: &c.

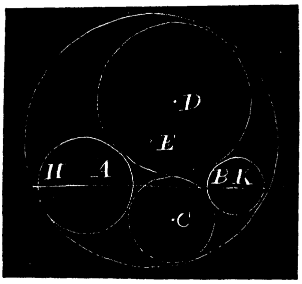
GENERAL SOLUTION OF PROBLEM ON P. 143, *VOL. V.*

BY E. B. SEITZ, GREENVILLE, OHIO.

Problem.—"It is required to describe upon the same plane, three circles touching each other, each of which shall touch two given circles."

Solution.—The two given circles may be wholly exterior to each other, or one may be wholly within the other. Hence there are two cases.

1. Suppose the construction completed, A, B being the given circles, and C, D, E the required circles. Now since each of the circles A, B touches the three circles C, D, E, which touch each other, the square of the common tangent of A, B is equal to four times the product of their diameters. A demonstration of this theorem has been given by Mr. Turrell in Vol. I, pp. 171–3. The following is a somewhat shorter demonst.



Let a, b, c, d be the radii of the circles A, B, C, D, and t_1, t_2, t_3, t_4, and T, T' the common tangents of A and C, C and B, B and D, D and A, A and B, and C and D.

Then, since A, C, B, D are all touched by the circle E in the same way, according to the principle stated on page 24, Vol. II, we have

$$TT' = t_1t_3 + t_2t_4.$$

But $t_1 = 2\sqrt{(ac)}$, $t_2 = 2\sqrt{(cb)}$, $t_3 = 2\sqrt{(bd)}$, $t_4 = 2\sqrt{(da)}$, and $T' = 2\sqrt{(cd)}$; $\therefore T \times 2\sqrt{(cd)} = 2\sqrt{(ac)} \times 2\sqrt{(bd)} + 2\sqrt{(cb)} \times 2\sqrt{(da)}$, whence $T^2 = 16ab = 4(2a)(2b)$; $\therefore AB^2 = (a-b)^2 + T^2 = a^2 + 14ab + b^2$; that is, the radii a, b being given, the distance between the centers of the circles A, B must be $\sqrt{(a^2+14ab+b^2)}$. We have then the following

Construction.—Take any radius r, not less than the half of HK. With A, B as centers and radii $r - a$ and $r - b$ describe arcs intersecting at E; then with center E and radius r describe a circle; it will touch A and B.

The construction is completed by describing the circles C, D, each tang't to the circles A, B, E.

This problem, as stated on page 167, Vol. V, has been solved in a number of ways, and it is not necessary to give a construction here.

The method given in Chauvenet's Geometry, p. 366, can be readily applied.

Proof.—We have $t_1 = 2\sqrt{(ac)}$, $t_2 = 2\sqrt{(cb)}$, $t_3 = 2\sqrt{(bd)}$, $t_4 = 2\sqrt{(da)}$, and $T = 4\sqrt{(ab)}$. Substituting in (1), and reducing, we find $T' = 2\sqrt{(cd)}$; therefore the circles C, D touch each other.

It appears then that the relative position of the two given circles must be as stated above, and that any number of solutions of the problem with this condition can be made.

The circle E may be described so as to lie wholly on the space between the two common tangents of A and B, these circles touching it externally. Then the circles C, D will touch each other beyond B from A.

2. Let the circle B be wholly within the circle A, and let us use the same notation as in the first case. [The figure, to illustrate this case, must be drawn so that the circles C, D, E touch A internally and B externally.]

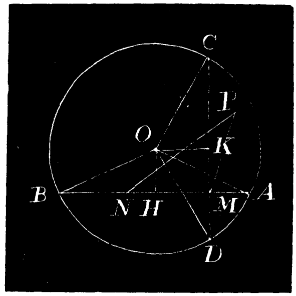
Now eq. (1) still holds, but t_1, t_4, T represent the common internal tangents of A and C, D and A, A and B, and are, therefore imaginary. We have $t_1 = 2\sqrt{(-ac)}$, $t_4 = 2\sqrt{(-da)}$, and t_2, t_3, T' as above. Hence (1) gives $T^2 = -16ab$. Therefore $AB^2 = (a + b)^2 + T^2 = a^2 - 14ab + b^2$; that is, the distance between the centers of the circles A, B must be $\sqrt{(a^2-14ab+b^2)}$. Moreover, b can not be greater than $(7-4\sqrt{3})a$.

The construction and proof are similar to those in the first case.

Problem 247. "Three points are taken at random on the surface of a circle; find the chance that the triangle formed by joining them is acute."

[By special request, we insert, below, the solution of this problem as submitted by the proposer, Mr. E. B. Seitz.]

Let M, N, P be the three random points, O the center of the circle, AB, the chord through M, N, and CD the chord through M perpendicular to AB. Draw OH and OK perpendicular to AB and CD.



Now N being between B and M, the angle M of the triangle MNP is obtuse if P lies in the segment CAD.

Let $OA=r$, $MN=x$, $\angle AOH=\theta$, $COK=\varphi$, and $\psi=$ the angle AB makes with some fixed line. Then $BH=r\sin\theta$, $OK=MH=r\sin\varphi$, area seg. $CAD=r^2(\varphi-\sin\varphi\cos\varphi)$; an element of the circle at M is $r\sin\theta\, d\theta\, r\sin\varphi\, d\varphi$, and at N it is $x\,dx\,d\psi$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $\frac{1}{2}\pi-\theta$ and $\frac{1}{2}\pi+\theta$; of x, 0 and $r(\sin\theta+\cos\varphi)=x'$, and doubled; and of ψ, 0 and 2π.

Hence, since the whole number of ways the three points can be taken is π^3r^6, the chance that the angle M is obtuse is

$$p_1=\frac{2}{\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta}\int_0^{x'}\int_0^{2\pi}r^2(\varphi-\sin\varphi\cos\varphi)r\sin\theta d\theta\, r\sin\varphi d\varphi\, xdx\, d\psi$$

$$=\frac{4}{\pi^2r^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta}\int_0^{x'}(\varphi-\sin\varphi\cos\varphi)\sin\theta\sin\varphi\, d\theta\, d\varphi\, xdx$$

$$=\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta}(\varphi-\sin\varphi\cos\varphi)(\sin\theta+\cos\varphi)^2\sin\theta\cos\varphi\, d\theta\, d\varphi$$

$$=\frac{1}{3\pi^2}\int_0^{\frac{1}{2}\pi}(8\pi\sin^2\theta+3\theta-12\theta\sin^2\theta-3\sin\theta\cos\theta-6\sin^3\theta\cos\theta)\sin^2\theta d\theta$$

$=\frac{3}{8}-\frac{4}{3\pi^2}$. Hence the chance that the triangle is obtuse is $3p_1$ $=\frac{9}{8}-\frac{4}{\pi^2}$, and the chance that it is acute is $p=1-3p_1=\frac{4}{\pi^2}-\frac{1}{8}$.

DEMONSTRATION OF A PROPOSITION.

BY THE EDITOR.

THE following *Proposition* has been sent to us by a subscriber with the request that a demonstration be given.

As the proposition, in the form here given is new to us, and as we have obtained a very elementary demonstration, assuming that it may interest *other* subscribers, also, we submit it to our readers.